On isomonodromic deformation problems on the torus associated with Lax pairs for elliptic integrable lattice systems

Frank Nijhoff
The University of Leeds, Shanghai University

I will introduce a class of Lax pairs for elliptic lattice equations comprising Q4 and a version of the lattice Landau-Lifschitz equation, and show how to deautonomise them to obtain isomonodromic deformation problems on the torus. I will derive the associated constitutive relations, and analyze them in the simplest cases to show they exhibit the behaviour of discrete Painleve type equations.

A holomorphic, rational matrix Lax pair for $\mathrm{q}-\mathrm{P_{VI}}$

Robert Conte
École normale suprieure de Paris-Saclay, The University of Hong Kong

The matrix Lax pairs of Jimbo and Miwa [4] for $\mathrm{P_{VI}}$ and of Jimbo and Sakai [5] for $\mathrm{q}-\mathrm{P_{VI}}$ do not exist when respectively $\theta_\infty^2=1$ and $\kappa_1=\kappa_2$, in their notation, i.e. when the residue at infinity, assumed diagonal, has a double eigenvalue.

For $\mathrm{P_{VI}}$, such a matrix Lax pair holomorphic in the four monodromy exponents is the moving frame of Bonnet surfaces [1, 3], the result being a Jordan matrix instead of a diagonal one for the residue at infinity.

For $\mathrm{q}-\mathrm{P_{VI}}$, by lack of discrete Bonnet surfaces, a Jordan matrix assumption allows us to obtain an explicit matrix Lax pair for any κ_1,κ_2, holomorphic in all the discrete monodromy exponents, rational in the two dependent variables of $\mathrm{q}-\mathrm{P_{VI}}$, whose residue at infinity is $\begin{pmatrix} A_{\infty,11} & 1 \\ 0 & A_{\infty,22} \end{pmatrix}$.

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Elliptic Garnier systems from 6d $\mathrm{Sp}(\mathrm{N})$ gauge theories

Xin Wang
University of Science and Technology of China

Supersymmetric gauge theories have close connections to integrable systems. For instance, one can explicitly construct supersymmetric defects in gauge theory, which act as eigenfunctions of certain differential or difference operators. In our work, we construct the $1/2$-BPS defect for 6d $\mathrm{Sp}(\mathrm{N})$ gauge theories and we find the corresponding operators are related to elliptic Garnier systems.

Initial value spaces of 4-dimensional Garnier systems

Tomoyuki Takenawa
Tokyo University of Marine Science and Technology

We will introduce how to construct initial value spaces in higher dimensions through the construction of initial value spaces of the 4-dimensional Garnier systems. Starting from discrete symmetries, we construct initial value spaces as compact rational varieties. Discrete symmetries act as pseudo-isomorphisms on initial value spaces and as isomorphisms on the Neron-Severi dual lattice.

Birational Weyl group actions and q-Painlevé equations via mutation combinatorics in cluster algebras

Teruhisa Tsuda
Aoyama Gakuin University

A cluster algebra is an algebraic structure generated by operations of a quiver (a directed graph) called the mutations and their associated simple birational mappings. By using a graph-combinatorial approach, we present a systematic way to derive a tropical, i.e. subtraction-free birational, representation of Weyl groups from cluster algebras. Our result provides a broad class of Weyl group actions including previously known examples acting on certain rational varieties and hence it is relevant to q-Painlevé equations and their higher-order extensions. Key ingredients of the argument are the combinatorial aspects of the reflection associated with a cycle subgraph in the quiver. We also discuss symplectic structures of the discrete dynamical systems thus obtained. This talk is based on a joint work with Tetsu Masuda and Naoto Okubo.

On the geometry of a 4-dimensional extension of the $A_1^{(1)}$ q-Painlevé I equation

Alexander Stokes
Waseda University

The Sakai scheme defines and classifies discrete Painlev equations in terms of associated generalised Halphen surfaces. However, there is not yet a similar framework for higher-dimensional analogues of discrete Painlevé equations. While a mechanism for associating rational varieties to such higher discrete Painlevé equations via a kind of regularisation of singularities is known, it is not yet clear what the general definition of higher-dimensional analogues of generalised Halphen surfaces should be - we need more examples to inform the development of a general theory. To this end, in this talk we report on the geometry of a 4-dimensional integrable mapping which extends the autonomous form of a q-Painlevé I equation with surface type $A_7^{(1)}$ and symmetry type $A_1^{(1)}$ in the Sakai classification. We construct a rational variety to which the mapping lifts to a pseudoautomorphism, obtained from $(\mathbb{P}^1)^4$ by blowing-up along 28 subvarieties. We deform this into a family of varieties which admits an action of an extended affine Weyl group by pseudo-automorphisms of the family and use it to obtain two 4-dimensional analogues of discrete Painlevé equations, one of which is a deautonomisation of the original autonomous mapping. Based on joint work with Adrian Stefan Carstea and Tomoyuki Takenawa

Discrete Painlevé equations from geometric deautonomization of QRT maps

Anton Dzhamay
BIMSA

In this talk we consider some examples of discrete Painlevé equations that can be obtained from a given QRT map using the technique of geometric deautonomization. One common interesting feature of such equations is that they often correspond to quasi-translations, or the elements of infinite order in the corresponding affine Weyl group whose certain power is a translation. Such elements often become translations if one considers a smaller affine Weyl subgroup, the phenomena that is known as the projective reduction.

Higher q-Painlevé equations from reduced cluster integrable systems

Andrei Marshakov
Skolkovo Institute of Science and Technology

I plan to start with the explanation how q-difference Painlevé equations and their generalizations arise from deautonomization of the Goncharov-Kenyon integrable systems on Poisson cluster varieties. Equivalent integrable systems are related by mutations in a dual cluster structure, and the q-Painlevé cases are distinguished to be self-dual in this picture. In order to complete the story and fill the gap with the higher q-Painlevé cases we need to extend the Goncharov-Kenyon class of integrable systems by their cluster Hamiltonian reductions.

Character varieties, DAHAs, and integrable systems

Oleg Chalykh
The University of Leeds

We study certain GL $(2n)$-character varieties of the 4 -punctured sphere that can be viewed as the monodromy spaces for higher Painlevé-VI. In rank 2 (for $n = 1$), these are a family of affine cubic surfaces well-known from Painlevé theory. We relate these varieties to DAHAs, and use this relation to interpret these varieties as natural phase spaces for the classical trigonometric Van Diejen's system. This is a joint work with Brad Ryan (Leeds), arxiv.org:2410.23456.

Combinatorics and large genus asymptotics of the Brézin-Gross-Witten numbers

Di Yang
University of Science and Technology of China

We discover and prove the uniform large genus leading asymptotics of certain normalized Brézin-Gross-Witten numbers, and give a new proof of the polynomiality phenomenon for the large genus. An application to the Painlevé II hierarchy is also obtained. The talk is based on a joint work with Jindong Guo, Paul Norbury and Don Zagier.

Painlevé and higher Painlevé equations in complex function theory

Rod Halburd
University College London

We discuss a natural class of value distribution problems for meromorphic functions and show that their solutions are characterised by Painlevé-type equations, including higher Painlev"| equations. These problems arise naturally when determining all meromorphic solutions of differential equations whose general solutions are not meromorphic and proving that all such solutions have been found. However, these value distribution results are also of independent interest.

On bilinear equations for irregular conformal blocks and quantum Painlevé tau functions

Hajime Nagoya
Kanazawa University

It is well known that the tau functions of Painlevé equations satisfy bilinear equations. We explain that bilinear equations for Virasoro irregular conformal blocks with the central charge $c = 1$naturally coincide with those of Painlevé equations. In general c, counterparts of irregular conformal blocks are bilinear equations for quantum Painlevé tau functions.

Cauchy bi-orthogonal polynomials and integrable lattices

Guofu Yu
Shanghai Jiaotong University

We first consider the generalised two-parameter Cauchy two-matrix model and the corresponding integrable lattice equation. In the second part, we construct discrete spectral transformations for Cauchy orthogonal polynomials, and find its corresponding discrete integrable systems. It turns out that the normalization factor of Cauchy orthogonal polynomials acts as the tau-function of the discrete CKP equation with a discrete Gram determinant structure.

A recent study on the CKP equation

Ying-ying Sun
University of Shanghai for Science and Technology

In this talk, I will show the generalized Cauchy matrix scheme for both the continuous and lattice CKP equation which can help us get their soliton solutions and some related exact solutions. Moreover, an elliptic direct linearization scheme is set up for the lattice CKP equation. As a consequence, the lattice CKP equation in the elliptic form together with its Lax pairs can be derived. By selecting appropriate measures and domains for the singular linear integral equation in the scheme, novel elliptic multi-soliton solutions of the lattice CKP equation are presented as well.

Deformed orthogonal functions and integrable lattices

Xiangke Chang
Chinese Academy of Sciences

Since the 1990s, the theory of orthogonal polynomials has been increasingly playing an important role in the studies of Toda type lattices, peakon dynamical systems of the Camassa-Holm type, as well as specific Painlevé equations. These integrable lattices can be derived according to deformations of orthogonal functions, directly or indirectly. This talk is devoted to exploring some of related works with focus on our recent results for some new orthogonality. In particular, some higher order discrete Painlevé-type equations are obtained.

D-modules approach to discrete special functions

Yik Man Chiang
The Hong Kong University of Science & Technology

It is well-known that D-modules techniques have been adopted to drive hypergeometric type identities. We show that it is also possible to use D-modules to describe classical special functions in general. Different manifestations of the "D" will then enable one to identify and perhaps to discover discrete analogues of classical special functions. Examples will be constructed to illustrate this idea.

The application of symmetric functions in universal character-type integrable systems

Zhaowen Yan
Inner Mongolia University

The universal character hierarchy is a generalization of the KP hierarchy. In this talk, we will discuss the tau functions of universal character-type integrable systems using the quantum field presentation of certain symmetric functions. Furthermore, we will present and prove the Pieri rules for the universal character and the symplectic universal character.

Isomonodromy equations and the inverse monodromy problem

Qian Tang
The University of Hong Kong

In this talk, we will introduce our recent work on some isomonodromy equations. We can regard it as a higher Painlevé equations. Our main result is the asymptotic behavior of its generic solution; we also provide its series solution starting from the monodromy data and use this method to study the inverse monodromy problem.

Asymptotics of Okubo system

Yuancheng Xie
Beijing University

Okubo system is the Birkhoff canonical form for differential equations of Poincare rank 1. It appears in the construction of Dubrovin's Frobenius manifold and also Bridgeland stability condition etc. Its isomonodromy deformation is an important source of higher rank Painlevé systems. In this talk I will outline a way to study the asymptotic behavior of its solutions at some boundary points using the Riemann-Hilbert techniques.

Painlevé equations, crystal basis, Riemann-Hilbert problems

Xiaomeng Xu
Beijing University

The Painlevé VI equation is equivalent to the isomonodromy deformation equation of meromorphic linear systems of rank 3 with a second order pole. Its boundary condition, monodromy formula and many global properties were given in 1980s. The isomonodromy equations of meromorphic linear systems of rank n with a second order pole can be seen as higher Painlevé equations. This talk recalls the long time asymptotics and monodromy formula for such higher Painlevé equations, and then explains how the analysis formula is related to various structures in the representation theory of quantum groups, like crystal basis, Gelfand-Tsetlin basis, and combinatorics of Young tableaux.

Integrable nonlocal finite-dimensional Hamiltonian systems related to the Ablowitz-Kaup-Newell-Segur system

Ruguang Zhou
Jiangsu Normal University

The method of nonlinearizing the Lax pair represents an efficient approach for decomposing $(1+1)$ -dimensional soliton equations into finite-dimensional integrable Hamiltonian systems. This method is useful for constructing the algebra-geometric solutions of soliton equations. In this talk, we will extend the method of nonlinearizing the Lax pair to the Ablowitz-Kaup-Newel-Segur (AKNS) equation under space-inverse reductions. We obtain a new class of finite-dimensional Hamiltonian systems. These systems are nonlocal, as they involve the inverse of the space variable. For these nonlocal Hamiltonian systems, we demonstrate that they maintain Liouville integrability and can be linearized on the Jacobi variety. Additionally, we show how to construct the algebro-geometric solutions of the AKNS equation with space-inverse reductions by means of our nonlocal finite-dimensional Hamiltonian systems. As an application, we obtain algebro-geometric solutions of the AKNS equation with Dirichlet and Neumann boundary conditions, as well as algebro-geometric solutions of the nonlocal nonlinear Schrödinger (NLS) equation.

On compatible ODE systems

Cheng Zhang
Shanghai University

Dubrovin constructed an infinite compatible ODE system that is equivalent to the KdV hierarchy. The construction is based on a r-matrix structure. In this talk, I will present some generalization of Dubrovins approach. The talk is based on a joint work with Di Yang, Zejun Zhou.

On the Painlevé II hierarchy and their applications

Lun Zhang
Fudan University

In this talk, I will report my recent works on the asymptotic aspects of the Painlevé II hierarchy. Their applications in random matrix theory and integrable differential equations will also be discussed.

A natural deformation of solutions for semi-Hamiltonian hydrodynamic type systems to a class of dispersive integrable systems

Maksim Pavlov
Shandong University of Science and Technology

In this talk we consider dispersive integrable systems and their dispersionless limits. In a dispersionless limit these systems are integrable by the Tsarev Generalized Hodograph Method, which determines a general solution. It was known that reflectionless potentials (multi-soliton solutions, rational solitons, multi-phase solutions, etc.) have no a dispersionless limit. In this talk we give the answer to the natural question for dispersive integrable systems: is it possible to find exact analytic solutions, which possess a dispersionless limit? Yes, this class also is well known in the theory of dispersive integrable systems: "isomonodromic deformations". We illustrate our construction by a set of such particular solutions determined by corresponding Lax representations. in 2D we consider the Kaup-Boussinesq system, while in 3D we investigate the Mikhalev system.

Darboux transformations for the KdV-type hierarchies

Zhiwei Wu
Sun Yat-sen University

The KdV-type hierarchies can be related to affine Kac-Moody algebras, and some of them have natural geometric correspondence. In this talk, we will discuss the Darboux transformations for KdV-type hierarchies in terms of loop group factorization. Permutability formulas are derived to construct explicit solutions. We will give some explicit examples, such as the Boussinesq and Kaup "CKupershmidt equations. They are associated to different types of affine Kac-Moody algebra and related to curve flows under certain group actions.